Laws of Change of Concepts

I. Space of Concepts

(incomplete working draft)

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Abstract

Motivated by the urgency of making explicit "the laws of possible rational passage from one concept to another" (Lawvere, 2011), here we begin to characterize the space of all mathematical concepts, which determines all possible passages between concepts.

Introduction

Developing a scientific understanding of concepts, beginning with concept formation (Lawvere, 2013a, pp. 7-8), is where cognitive science failed (Fodor, 1998a, b, 2006) and continues to do so (Núñez et al., 2019), with weird pride in their ignorance that's beyond the reach of commonsense (cf. Pinker, 2005). So, we ignore the so-called "cognitive science", which refuses to learn and apply the required, albeit advanced, mathematical methods (Lawvere, 1994a, 1999, p. 412, 2013b). Here we characterize the space of concepts using time-tested methods of science dating back to Galileo (see Lawvere and Schanuel, 2009, pp. 3-9, 308-309), along with the above noted advances in mathematics involving category theory.

What is a concept?

A concept is a collection or a set of properties, according to Frege (ibid., p. 380). Working mathematicians, having recognized its inadequacy (Lawvere, 2013, p. 220), developed a more refined understanding of concept, which includes not only properties but also their mutual determinations. Broadly speaking, with properties and their determinations as objects and morphisms, respectively, the abstract general that is a concept becomes a mathematical category (see Appendix A1 in Posina, Ghista, and Roy, 2017 for a brief account of mathematical category readily accessible to total beginners). To help everyone appreciate the scientific revolution that is the above mathematical account of concepts (to be elaborated substantially in the following sections), prior to F. William Lawvere's Functorial Semantics of Algebraic Theories (Lawvere, 1963), Bastiani and Ehresmann's Sketches (Bastiani and Ehresmann, 1972), and Grothendieck's Descent (see Clementino and Picado, 2007/2008, p. 15), a theory of a category of mathematical objects, say, graphs was not a mathematical object: one had to leave mathematical universe(s) of discourse (cf. graphs) and go to an alien, relatively speaking, universe of language (e.g., English) in order to present its theory (of graphs): graphs consist of dots and arrows; every arrow has a source and a target dot. Thanks to functorial semantics, sketches, and descent, a theory (abstract general / concept) of a category of mathematical objects is a mathematical object (a category). More explicitly, the theory of graphs is a graph (consisting of two objects: a dot D and an arrow A, along with two morphisms: *source*, *target*: D ---> A, which include the dot into the arrow as source and target dots; see Figures 1-5 in Posina, Ghista, and Roy, 2017 for additional examples of categories and their theories, including this theory of the category of graphs; see also Lawvere, 1966, 1972, pp. 89-152, 1994b, 2004, pp. 10-13, 2002, pp. 267-269, 2003, pp. 215, 217-219, 2007, p. 46, 2008, pp. 6-7, 2011, p. 252; Lawvere and Rosebrugh, 2003, pp. ix-x, 154-155, 235-236, 245-247; Lawvere and Schanuel, 2009, pp. 149-151, 269-270).

Space of concepts

How are we going to construct the space of all mathematical concepts? With concepts as mathematical categories, the category of categories (Lawvere, 1966) can be considered as the space of concepts. With concepts as categories, all possible passages from one concept to another would be functorial (i.e., functors between the corresponding categories; see Appendix A2, Posina, Ghista, and Roy, 2017 for a readily accessible account of functors; see Lawvere and Rosebrugh, 2003, pp. 96-135 for examples of functoriality i.e., compatibility with composition).

A more definite account of the space of concepts can be based on the fact: “All Concepts Are Kan Extensions” (Mac Lane, 1998, pp. 248-250; see also Lawvere, 2004, pp. 8-13), thanks to which the totality of all possible Kan extensions constitutes the space of concepts. We can then compare and contrast the totality of Kan extensions and the category of categories with the objective of abstracting the laws of all possible rational passages between concepts and advancing the long overdue science of proper thinking. To be clear, we don't consider the present manuscript the last word in any sense for the simple reason that generalizations of Kan extension (see Ehresmann, 1984, pp. 20-30) and mathematical studies of concepts within the framework of Memory Evolutive Systems (Ehresmann and Vanbremeersch, 2007) are two significant scientific advances of great relevance in the present context, which we plan to discuss in our subsequent investigations of thinking.

What is an extension?

The above alluded universality of Kan extensions was explicitly spelled out in F. William Lawvere's functorial semantics of algebraic theories (Lawvere, 1963, 2004, p. 8), and can be understood as resulting from the fact that all mathematical concepts and constructions (e.g., product, truth value object) are in terms of universal properties, i.e., with respect to the entire universe of discourse (Lawvere and Schanuel, 2009, p. 213), or equivalently in terms of 'good for' (X is what X is good for, which is a refinement of functional definitions; see Lawvere and Rosebrugh, 2003, pp. 26-27; Lawvere and Schanuel, 2009, p. 334).

Before we discuss Kan extension, let us begin with addressing the title question (of this section). First, let us consider a function:

$$f\colon A \dashrightarrow B$$

There are two dialectically opposed readings of the notion of function:

1. Geometric: f is an A-shaped figure in B
2. Algebraic: f is a B-valued property of A.

Unlike contemporary post-modern "philosophical" readings of mathematical advances, especially that of category theory, the above contrast is consequential (Lawvere and Schanuel, 2009, pp. 81-85, 370-371; see also Lawvere and Rosebrugh, 2003, pp. 171-176). Given that properties and their mutual determinations figure prominently in our above alluded understanding of concepts (as mathematical categories or as Kan extensions), our approach is algebraic in the beginning and ends up geometric as we get to the space of concepts. (It may be noted that discussion of a related philosophically profound dual: extensive *vs.* intensive, the mathematical significance of which was recognized by Grassmann (see Lawvere, 1992, 2002, 2003, 2007, 2015), will be presented in our subsequent manuscript.)

Let us consider two properties of an object (of a category of objects) A:

$$f\colon A \dashrightarrow B$$

and

$$g\colon A \dashrightarrow C$$

where codomain objects B and C are property types (e.g., numbers). One immediate question, all too familiar in scientific investigations is, say, does the property f of A *determine* the property g (Lawvere and Schanuel, 2009, pp. 68-70)? Given a map

$$h\colon B \dashrightarrow C$$

such that

$$g = h \circ f$$

(where ‘o’ denotes composition) we say *f* determines *g* (or *g* is determined by *f*), with *h* as the proof of determination, which is also known as *extension*. More explicitly, *h* is an extension of *g* along *f* (Lawvere and Rosebrugh, 2003, pp. 37-38). Since composition is a non-commutative multiplication, the determination of properties or finding extensions is a division problem involving ratios (Lawvere, 2002, pp. 268-269, 2011, p. 250; Lawvere and Schanuel, 2009, p. 45, which is discussed in detail in Appendix I).

Adjoints of induced functors

Kan extensions are adjoints of induced functors (Lawvere, 2004, pp. 8-13). A map

$$h\colon \mathrm{B} \dashrightarrow \mathrm{C}$$

from a property type B to a property type C *induces* a map

$$h^{A}\colon \mathrm{B}^{\mathrm{A}} \dashrightarrow \mathrm{C}^{\mathrm{A}}$$

from the map object of all B-valued properties of A to the map object of all C-valued properties of A, and is defined as:

$$h^{A}(f) = h \circ f$$

for each property

$$f\colon \mathrm{A} \dashrightarrow \mathrm{B}$$

in the domain map object B^{A}.

With objects as categories, we obtain induced functors. A functor H from one codomain category ($\boldsymbol{B}$) to another codomain category ($\boldsymbol{C}$) of a pair of functors with a common domain category ($\boldsymbol{A}$)

$$H: \boldsymbol{B} \dashrightarrow \boldsymbol{C}$$

induces a covariant functor

$$H^{\boldsymbol{A}}: \boldsymbol{B}^{\boldsymbol{A}} \dashrightarrow \boldsymbol{C}^{\boldsymbol{A}}$$

between the corresponding functor categories, and defined as:

$$H^{\mathbf{A}} (F) = H \circ F$$

for every functor

$$F: \boldsymbol{A} \dashrightarrow \boldsymbol{B}$$

in the domain functor category $\boldsymbol{B}^{\boldsymbol{A}}$.

Dually, a functor

$$Y: \boldsymbol{P} \dashrightarrow \boldsymbol{Q}$$

from the domain category ($\boldsymbol{P}$) to the domain category ($\boldsymbol{Q}$) of a pair of functors

$$W: \boldsymbol{P} \dashrightarrow \boldsymbol{R}$$

and

$$Z: \boldsymbol{Q} \dashrightarrow \boldsymbol{R}$$

with a common codomain category ($\boldsymbol{R}$), which can be considered as $\boldsymbol{R}$-valued properties of $\boldsymbol{P}$ and $\boldsymbol{Q}$, respectively, induces a contravariant functor

$$\boldsymbol{R}^{Y}\colon \boldsymbol{R}^{\boldsymbol{Q}} \text{ ---> } \boldsymbol{R}^{\boldsymbol{P}}$$

defined as

$$\boldsymbol{R}^{Y}(Z) = Z \text{ o } Y$$

for every functor

$$Z\colon \boldsymbol{Q} \text{ ---> } \boldsymbol{R}$$

in the domain functor category $\boldsymbol{R}^{\boldsymbol{Q}}$.

Let us now consider adjoints of a functor induced by a functor between two distinct (familiar) domain categories

$$L\colon \boldsymbol{I} \text{ ---> } \boldsymbol{2}$$

of two Set-valued contravariant functors:

$$F\colon \boldsymbol{2} \text{ ---> } \boldsymbol{S}$$

$$G\colon \boldsymbol{I} \text{ ---> } \boldsymbol{S}$$

where ***2*** is a category consisting of two objects and one (non-identity) morphism (Lawvere and Rosebrugh, 2003, pp. 114-119), ***I*** is a category consisting of two parallel morphisms between two objects (Lawvere and Schanuel, 2009, p. 150), and ***S*** is the category of sets. The functor

$$L\colon \boldsymbol{I} \text{ ---> } \boldsymbol{2}$$

induces an inclusion (contravariant) functor

$$\boldsymbol{S}^{L}\colon \boldsymbol{S}^{\boldsymbol{2}} \text{ ---> } \boldsymbol{S}^{\boldsymbol{I}}$$

from the domain category of functions to the codomain category of irreflexive directed graphs; here the category of functions is included as the subcategory of loops (with the one function serving as both source and target functions) of the codomain category of graphs (Lawvere and Schanuel, 2009, p. 144). This inclusion of functions as loops (arrows with target dot = source dot) has both left and right adjoints, which are Kan extensions (Lawvere, 1972, p. 133; see Appendix II for calculation of adjoints of induced functors). The totality of all possible Kan extensions constitutes the space of concepts, which determines the laws of rational passage between concepts.